

Born–Jordan Quantization and the Uncertainty Principle

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Abstract

The Weyl correspondence and the related Wigner formalism lie at the core of traditional quantum mechanics. We discuss here an alternative quantization scheme, whose idea goes back to Born and Jordan, and which has recently been revived in another context, namely time-frequency analysis. We show that in particular the uncertainty principle does not enjoy full symplectic covariance properties in the Born and Jordan scheme, as opposed to what happens in the Weyl quantization.

1 Introduction

The problem of “quantization” of an “observable” harks back to the early days of quantum theory; mathematically speaking, and to use a modern language, it is the problem of assigning to a symbol a pseudo-differential operator in a way which is consistent with certain requirements (symmetries under a group of transformations, positivity, etc.). Two of the most popular quantization schemes are the Kohn–Nirenberg and Weyl correspondences. The first is widely used in the theory of partial differential equations and in time-frequency analysis (mainly for numerical reasons), the second is the traditional quantization used in quantum mechanics. Both are actually particular cases of Shubin’s pseudo-differential calculus, where one can associate to a given symbol a an infinite family $(A_\tau)_\tau$ of pseudo-differential operators parametrized by a real number τ , the cases $\tau = 1$ and $\tau = \frac{1}{2}$ corresponding to, respectively, Kohn–Nirenberg and Weyl operators. It turns

out that each of Shubin's τ -operators can be alternatively defined in terms of a generalization Wig_τ of the usual Wigner distribution by the formula

$$\langle A_\tau \psi, \bar{\phi} \rangle = \langle a, \text{Wig}_\tau(\psi, \phi) \rangle$$

and this observation has recently been used by researchers in time-frequency analysis to obtain more realistic phase-space distributions (more about this in the discussion at the end of the paper). They actually went one step further by introducing a new distribution by averaging Wig_τ for the values of τ in the interval $[0, 1]$. This leads, via the analogue of the formula above to a third class of pseudo-differential operators, corresponding to the averaging of Shubin's operators A_τ . It was noted by the present author that this averaged pseudo-differential calculus is actually an extension of one of the first quantization schemes discovered by Born, Jordan, and Heisenberg around 1927, prior to that of Weyl's.

The aim of the present paper is to give a detailed comparative study of the Weyl and Born–Jordan correspondences (or “quantization schemes”) with an emphasis on the symplectic covariance properties of the associated uncertainty principles.

Notation 1 *We write $x = (x_1, \dots, x_n)$ and $p = (p_1, \dots, p_n)$ and $z = (x, p)$. In matrix calculations x, p, z are viewed as column vectors. The phase space $\mathbb{R}^{2n} \equiv \mathbb{R}^n \times \mathbb{R}^n$ is equipped with the standard symplectic form $\sigma(z, z') = px' - p'x$; equivalently $\sigma(z, z') = Jz \cdot z'$ where $J = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{pmatrix}$ is the standard symplectic matrix. We denote by $\mathcal{S}(\mathbb{R}^{2n})$ the Schwartz space of rapidly decreasing smooth functions and by $\mathcal{S}'(\mathbb{R}^{2n})$ its dual (the tempered distributions).*

2 Discussion of Quantization

After Werner Heisenberg's seminal 1925 paper [23] which gave rigorous bases to the newly born “quantum mechanics”, Born and Jordan [4] wrote the first comprehensive exposition on matrix mechanics, followed by an article with Heisenberg himself [5]. These articles were an attempt to solve an ordering problem: assume that some quantization process associated to the canonical variables x (position) and p (momentum) two operators \hat{X} and \hat{P} satisfying the canonical commutation rule $\hat{X}\hat{P} - \hat{P}\hat{X} = i\hbar$. What should then the operator associated to the monomial $x^m p^n$ be? Born and Jordan's answer

was

$$x^m p^n \xleftrightarrow{\text{BJ}} \frac{1}{n+1} \sum_{k=0}^n \hat{P}^{n-k} \hat{X}^m \hat{P}^k \quad (1)$$

which immediately leads to the “symmetrized” operator $\frac{1}{2}(\hat{X}\hat{P} + \hat{P}\hat{X})$ when the product is xp . In fact Weyl and Born–Jordan quantization lead to the same operators for all powers x^m or p^n , or for the product xp (for a detailed analysis of Born and Jordan’s derivation see Fedak and Prentis [9], also Castellani [6] and Crehan [7]). Approximately at the same time Hermann Weyl had started to develop his ideas of how to quantize the observables of a physical system, and communicated them to Max Born and Pascual Jordan (see Scholz [28]). His basic ideas of a group theoretical approach were published two years later [33, 34]. One very interesting novelty in Weyl’s approach was that he proposed to associate to an observable of a physical system what we would call today a Fourier integral operator. In fact, writing the observable as an inverse Fourier transform

$$a(x, p) = \int_{\mathbb{R}^{2n}} e^{i(ps+xt)} \mathcal{F}a(s, t) ds dt \quad (2)$$

he defined its operator analogue by

$$A = \int_{\mathbb{R}^{2n}} e^{i(\hat{P}s + \hat{X}t)} \mathcal{F}a(s, t) ds dt \quad (3)$$

which is essentially the modern definition that will be given below (formula (9)). We will denote the Weyl correspondence by $a \xleftrightarrow{\text{Weyl}} A_W$ or $A_W = \text{Op}(a)$. Weyl was led to this choice because of the immediate ordering problems that occurred when one considered other observables than monomials $a(x, p) = x^k$ or $a(x, p) = p^\ell$. For instance, using Schrödinger’s rule what should the operator associated with $a(x, p) = xp$ be? Weyl’s rule immediately yields the symmetrized quantization rule

$$a(\hat{X}, \hat{P}) = \frac{1}{2}(\hat{X}\hat{P} + \hat{P}\hat{X})$$

and one finds that more generally (McCoy [27], 1932)

$$x^m p^n \xleftrightarrow{\text{Weyl}} \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \hat{P}^{n-k} \hat{X}^m \hat{P}^k. \quad (4)$$

It turns out that the Weyl quantization rule (4) for monomials is a particular case of the so-called “ τ -ordering”: for any real number τ one defines

$$x^m p^n \xleftrightarrow{\tau} \sum_{k=0}^n \binom{n}{k} (1-\tau)^k \tau^{n-k} \hat{P}^k \hat{X}^n \hat{P}^{n-k} \quad (5)$$

this rule reduces to Weyl's prescription when $\tau = \frac{1}{2}$. When $\tau = 1$ one gets the “normal ordering” $\hat{X}^n \hat{P}^n$ familiar from the elementary theory of partial differential equations, and $\tau = 0$ yields the “anti-normal ordering” $\hat{P}^n \hat{X}^n$ sometimes used in physics. We now make the following fundamental observation: the Born–Jordan prescription (1) is obtained by averaging the τ -ordering on the interval $[0, 1]$ (de Gosson [18], de Gosson and Luef [21]). In fact

$$\int_0^1 (1-\tau)^k \tau^{n-k} d\tau = \frac{k!(n-k)!}{(n+1)!}$$

and hence

$$x^m p^n \xrightarrow{\text{BJ}} \frac{1}{n+1} \sum_{k=0}^m \hat{P}^k \hat{X}^n \hat{P}^{n-k}. \quad (6)$$

One interesting feature of the quantization rules above is the following: suppose that the operators \hat{X} and \hat{P} are such that

$$[\hat{X}, \hat{P}] = \hat{X}\hat{P} - \hat{P}\hat{X} = i\hbar.$$

then $[\hat{X}^m, \hat{P}^n]$ is independent of the choice of quantization; in fact (see Crehan [7] and the references therein):

$$[\hat{X}^m, \hat{P}^n] = \sum_{k=1}^{\min(m,n)} (i\hbar)^k \binom{m}{k} \binom{n}{k} \hat{P}^{n-k} \hat{X}^{m-k}. \quad (7)$$

In physics as well as in mathematics, the question of a “good” choice of quantization is more than just academic. For instance, different choices may lead to different spectral properties. The following example is due to Crehan [7]. Consider the Hamiltonian function

$$H(z) = \frac{1}{2}(p^2 + x^2) + \lambda(p^2 + x^2)^3.$$

The term that gives an ordering problem is evidently $(p^2 + x^2)^3$; Crehan then shows that the most general quantization invariant under the symplectic transformation $(x, p) \mapsto (p, -x)$ is

$$\hat{H} = \frac{1}{2}(\hat{P}^2 + \hat{X}^2) + \lambda(\hat{P}^2 + \hat{X}^2)^3 + \lambda(3\alpha\hbar^2 - 4)(\hat{P}^2 + \hat{X}^2).$$

The eigenfunctions of \hat{H} are those of the harmonic oscillator, and the corresponding eigenvalues are the numbers

$$E_N = (N + \frac{1}{2})\hbar + \lambda\hbar(2N + 1)^3 + \lambda\hbar(2N + 1)(3\alpha\hbar^2 - 4)$$

($N = 0, 1, 2, \dots$) which clearly shows the dependence of the spectrum on the parameters α and λ , and hence of the chosen quantization.

3 Born–Jordan Quantization

3.1 First definition

Let $z_0 = (x_0, p_0)$ and consider the “displacement” Hamiltonian function $H_{z_0} = \sigma(z, z_0)$. The flow determined by the corresponding Hamilton equations is given by $f_t(z) = z + tz_0$; for $\Psi \in \mathcal{S}'(\mathbb{R}^{2n})$ we define $T(z_0)\Psi(z) = (f_1)^*\Psi(z) = \Psi(z - z_0)$. The τ -quantization of H_{z_0} is the operator $\hat{H}_{z_0} = \sigma(\hat{Z}, z_0)$, $\hat{Z} = (\hat{X}, \hat{P})$; the solution of the corresponding Schrödinger equation at time $t = 1$ with initial condition ψ is given by the Heisenberg operator $\hat{T}(z_0) = e^{\frac{i}{\hbar}\sigma(\hat{Z}, z_0)}$; its action on $\psi \in \mathcal{S}'(\mathbb{R}^n)$ is explicitly given by

$$\hat{T}(z_0)\psi(x) = e^{\frac{i}{\hbar}(p_0x_0 - \frac{1}{2}p_0x_0)}\psi(x - x_0). \quad (8)$$

Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ be an observable (or “symbol”). By definition, the Weyl correspondence $a \xleftrightarrow{\text{Weyl}} A_W$ is defined by

$$A_W\psi = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z) \hat{T}(z) \psi dz \quad (9)$$

where $a_\sigma = \mathcal{F}_\sigma a$ is the symplectic Fourier transform of a , that is

$$a_\sigma(z) = \left(\frac{1}{2\pi\hbar}\right)^n \langle e^{-\frac{i}{\hbar}\sigma(z, \cdot)}, \cdot \rangle; \quad (10)$$

for $a \in \mathcal{S}(\mathbb{R}^{2n})$; informally

$$a_\sigma(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z, z')} a(z') dz'. \quad (11)$$

The symplectic Fourier transform \mathcal{F}_σ is an involution ($\mathcal{F}_\sigma^2 = I_d$); it is related to the usual Fourier transform \mathcal{F} on \mathbb{R}^{2n} by the formula $\mathcal{F}_\sigma a(z) = \mathcal{F}a(Jz)$, J the standard symplectic matrix. The action of the operator A_W on a function $\psi \in \mathcal{S}'(\mathbb{R}^n)$ is given by

$$A_W\psi = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z_0) \hat{T}(z_0) \psi dz_0. \quad (12)$$

How can we modify this formula to define Born–Jordan quantization? An apparently easy answer would be to first define τ -quantization by replacing \widehat{H}_{z_0} by its τ -quantized version $\widehat{H}_{z_0, \tau}$, and then to average the associated operators $\widehat{T}_\tau(z_0)$ thus obtained to get a “ $\widehat{T}_{\text{BJ}}(z_0)$ operator” which would allow to define \widehat{A}_{BJ} . However, such a procedure trivially fails, because all τ -quantizations of the displacement Hamiltonian H_{z_0} coincide with \widehat{H}_{z_0} as can be verified using the polynomial rule (5). There is however a simple way out of this difficulty; it consists in replacing, as we did in [21], $\widehat{T}(z_0)$ by $\Theta(z_0)\widehat{T}(z_0)$ where

$$\Theta(z_0) = \frac{\sin(p_0 x_0 / 2\hbar)}{p_0 x_0 / 2\hbar} \quad (13)$$

and we define the Born–Jordan operator A_{BJ} by

$$A_{\text{BJ}}\psi = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z) \Theta(z) \widehat{T}(z_0) \psi dz. \quad (14)$$

This formula will be justified below.

3.2 Pseudo-differential formulation

There is another way to describe Born–Jordan quantization. Writing formula (12) in pseudo-differential form yields the usual formal expression

$$A_{\text{W}}\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}p(x-y)} a\left(\frac{1}{2}(x+y), p\right) \psi(y) dp dy \quad (15)$$

for the Weyl correspondence (we assume for simplicity that $a \in \mathcal{S}(\mathbb{R}^{2n})$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$). We now define the τ -dependent operator *à la* Shubin [30]:

$$A_\tau\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}p(x-y)} a(\tau x + (1-\tau)y, p) \psi(y) dp dy; \quad (16)$$

the Born–Jordan operator A_{BJ} with symbol a is then defined by the average

$$A_{\text{BJ}}\psi = \int_0^1 A_\tau\psi d\tau \quad (17)$$

which we can write, interchanging the order of the integrations,

$$A_{\text{BJ}}\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}p(x-y)} a_{\text{BJ}}(x, y, p) \psi(y) dp dy \quad (18)$$

where

$$a_{\text{BJ}}(x, y, p) = \int_0^1 a(\tau x + (1-\tau)y, p) d\tau. \quad (19)$$

We have been a little bit sloppy in writing the (usually divergent) integrals above, but all three definitions become rigorous if we view the operators A_W , A_τ , and A_{BJ} as being defined by the distributional kernels

$$K(x, y) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} (\mathcal{F}_2^{-1}a)\left(\frac{1}{2}(x+y), p\right) \quad (20)$$

$$K_\tau(x, y) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} (\mathcal{F}_2^{-1}a)((\tau x + 1 - \tau)y, p) \quad (21)$$

(\mathcal{F}_2^{-1} is the inverse partial Fourier transform with respect to the second set of variables) and

$$K_{BJ}(x, y) = \int_0^1 K_\tau(x, y) d\tau \quad (22)$$

where \mathcal{F}_2^{-1} is the inverse Fourier transform in the second set of variables. We will give below an alternative rigorous definition, but let us first check that definition (18)–(22) coincides with the one given in previous subsection. Define the modified Heisenberg–Weyl operators

$$\hat{T}_\tau(z_0)\psi(x) = e^{\frac{i}{2\hbar}(2\tau-1)p_0x_0}\hat{T}(z_0)\psi(x) \quad (23)$$

that is

$$\hat{T}_\tau(z_0)\psi(x) = e^{\frac{i}{\hbar}(p_0x - (1-\tau)p_0x_0)}\psi(x - x_0). \quad (24)$$

These obey the same commutation rules

$$\hat{T}_\tau(z_0)\hat{T}_\tau(z_1) = e^{\frac{i}{\hbar}\sigma(z_0, z_1)}\hat{T}_\tau(z_1)\hat{T}_\tau(z_0) \quad (25)$$

as the usual Heisenberg operators $\hat{T}(z_0)$.

Proposition 2 *Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$, $\psi \in \mathcal{S}(\mathbb{R}^n)$. The Born–Jordan operator (17) is given by formula (14), that is*

$$A_{BJ}\psi = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z) \hat{T}_{BJ}(z) \psi dz \quad (26)$$

with

$$\hat{T}_{BJ}(z) = \Theta(z) \hat{T}(z) \quad , \quad \Theta(z) = \frac{\sin(px/2\hbar)}{px/2\hbar}. \quad (27)$$

In particular, A_{BJ} is the Weyl operator with symbol

$$a_{BJ} = \left(\frac{1}{2\pi\hbar}\right)^n a * \mathcal{F}_\sigma \Theta \quad (28)$$

Proof. (Cf. [21, 18]). One verifies by a straightforward computation that the Shubin formula (16) can be rewritten as

$$A_\tau \psi = \int_{\mathbb{R}^{2n}} a_\sigma(z) \widehat{T}_\tau(z) \psi dz. \quad (29)$$

Let us now average in τ over the interval $[0, 1]$; interchanging the order of integrations and using the trivial identity

$$\int_0^1 e^{\frac{i}{2\hbar}(2\tau-1)px} d\tau = \frac{2\hbar}{px} \sin \frac{px}{2\hbar}$$

we get

$$\widehat{T}_{\text{BJ}}(z) = \int_0^1 \widehat{T}_\tau(z) d\tau = \Theta(z) \widehat{T}(z)$$

hence formula (26). To prove the last statement we note that formula (26) can be rewritten

$$\begin{aligned} A_{\text{BJ}} \psi &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z) \Theta(z) \widehat{T}(z) \psi dz \\ &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} (a_{\text{BJ}})_\sigma(z) \widehat{T}(z) \psi dz \end{aligned}$$

where $(a_{\text{BJ}})_\sigma = a\Theta$. Taking the inverse Fourier transform we get, noting that $\mathcal{F}_\sigma(a * b) = (2\pi\hbar)^n \mathcal{F}_\sigma a \mathcal{F}_\sigma b$,

$$b = \left(\frac{1}{2\pi\hbar}\right)^n a * \mathcal{F}_\sigma^{-1} \Theta$$

hence (28) since $\mathcal{F}_\sigma^{-1} \Theta = \mathcal{F}_\sigma \Theta$ because the function Θ is even. ■

One easily verifies that the (formal) adjoint of $A_\tau = \text{Op}_\tau(a)$ is given by

$$\text{Op}_\tau(a)^* = \text{Op}_{1-\tau}(\overline{a}) \quad (30)$$

and hence

$$\text{Op}_{\text{BJ}}(a)^* = \text{Op}_{\text{BJ}}(\overline{a}). \quad (31)$$

Born–Jordan operators thus share with Weyl operators the property of being (essentially) self-adjoint if and only if their symbol is real. This property makes Born–Jordan prescription a good candidate for physical quantization, while Shubin quantization should be rejected being unphysical for $\tau \neq \frac{1}{2}$.

4 The Born–Jordan–Wigner Distribution

4.1 The τ -Wigner distribution

In a recent series of papers Boggiatto and his collaborators [1, 2, 3] have introduced a τ -dependent Wigner distribution $\text{Wig}_\tau(f, g)$ which they average over the values of τ in the interval $[0, 1]$. This procedure leads to an element of the Cohen class [22], *i.e.* to a transform of the type $C(\psi, \phi) = \text{Wig}_\tau(\psi, \phi) * \theta$ where $\theta \in \mathcal{S}'(\mathbb{R}^{2n})$. From the point of view of time-frequency analysis this can be interpreted as the application of a filter to the Wigner transform.

Let us define the τ -Wigner cross-distribution $\text{Wig}_\tau(\psi, \phi)$ of a pair (ψ, ϕ) of functions in $\mathcal{S}(\mathbb{R}^n)$:

$$\text{Wig}_\tau(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}py} \psi(x + \tau y) \overline{\phi}(x - (1 - \tau)y) dy. \quad (32)$$

Choosing $\tau = \frac{1}{2}$ one recovers the usual cross-Wigner transform

$$\text{Wig}(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}py} \psi(x + \frac{1}{2}y) \overline{\phi}(x - \frac{1}{2}y) dy \quad (33)$$

and when $\tau = 0$ we get the Rihaczek–Kirkwood distribution

$$R(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{-\frac{i}{\hbar}px} \psi(x) \overline{\mathcal{F}\phi}(p) \quad (34)$$

well-known from time-frequency analysis [22].

The mapping Wig_τ is a bilinear and continuous mapping $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$. When $\psi = \phi$ one writes $\text{Wig}_\tau(\psi, \psi) = \text{Wig}_\tau \psi$; it is the τ -Wigner distribution considered by Boggiatto *et al.* [1, 2, 3]). It follows from the definition of Wig_τ that we have

$$\overline{\text{Wig}_\tau(\phi, \psi)} = \text{Wig}_{1-\tau}(\psi, \phi); \quad (35)$$

in particular

$$\overline{\text{Wig}_\tau \psi} = \text{Wig}_{1-\tau} \psi \quad (36)$$

hence $\text{Wig}_\tau \psi$ is not a real function in general if $\tau \neq \frac{1}{2}$.

Proposition 3 *Assume that $\psi, \phi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} \text{Wig}_\tau(\psi, \phi)(z) dp = \psi(x) \overline{\phi}(x) \quad (37)$$

and

$$\int_{\mathbb{R}^n} \text{Wig}_\tau(\psi, \phi)(z) dx = \mathcal{F}\psi(p) \overline{\mathcal{F}\phi(p)}. \quad (38)$$

Proof. Formula (37) is straightforward. On the other hand

$$\int_{\mathbb{R}^n} \text{Wig}_\tau(\psi, \phi)(z) dx = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}py} \psi(x + \tau y) \overline{\phi}(x - (1 - \tau)y) dx dy$$

and setting $x' = x + \tau y$, $x'' = x - (1 - \tau)y$ we have $dx' dx'' = dx dy$ so that

$$\int_{\mathbb{R}^n} \text{Wig}_\tau(\psi, \phi)(z) dx = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}px'} \psi(x') e^{\frac{i}{\hbar}px''} \overline{\phi}(x'') dx dy$$

hence formula (38). Notice that the right-hand sides of (37) and (38) are independent of the parameter τ . ■

In particular [1], the τ -Wigner distribution $\text{Wig}_\tau \psi = \text{Wig}_\tau(\psi, \psi)$ satisfies the usual marginal properties:

$$\int_{\mathbb{R}^n} \text{Wig}_\tau \psi(z) dp = |\psi(x)|^2, \quad \int_{\mathbb{R}^n} \text{Wig}_\tau \psi(z) dx = |\mathcal{F}\psi(p)|^2. \quad (39)$$

There is a fundamental relation between Weyl pseudo-differential operators and the cross-Wigner transform, that relation is often used to define the Weyl operator $A_W = \text{Op}_W(a)$:

$$\langle A_W \psi | \overline{\phi} \rangle = \langle a, \text{Wig}(\psi, \phi) \rangle \quad (40)$$

for $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$. Not very surprisingly this formula extends to the case of τ -operators:

Proposition 4 *Let $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$, $a \in \mathcal{S}(\mathbb{R}^{2n})$, and τ a real number. We have*

$$\langle A_\tau \psi | \phi \rangle = \langle a, \text{Wig}_\tau(\psi, \phi) \rangle \quad (41)$$

where $\langle \cdot, \cdot \rangle$ is the distributional bracket on \mathbb{R}^{2n} and $A_\tau = \text{Op}_\tau(a)$.

Proof. By definition of Wig_τ we have

$$\begin{aligned} \langle a, \text{Wig}_\tau(\psi, \phi) \rangle &= \\ &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{3n}} e^{-\frac{i}{\hbar}py} a(z) \psi(x + \tau y) \overline{\phi}(x - (1 - \tau)y) dy dp dx. \end{aligned}$$

Defining new variables $x' = x - (1 - \tau)y$ and $y' = x + \tau y$ we have $y = y' - x'$, $dy dx = dy' dx'$ and hence

$$\begin{aligned} \langle a, \text{Wig}_\tau(\psi, \phi) \rangle &= \\ &= \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{3n}} e^{-\frac{i}{\hbar}p(x' - y')} a(\tau x' + (1 - \tau)y', p) \psi(y') \overline{\phi}(x') dy' dp dx'; \end{aligned}$$

the equality (41) follows in view of definition (16) of A_τ . ■

Formula (41) allows us to define $\hat{A}_\tau \psi = \text{Op}_\tau(a)\psi$ for arbitrary symbols $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ in the same way as is done for Weyl pseudo-differential operators: choose $\phi \in \mathcal{S}(\mathbb{R}^n)$; then $\text{Wig}_\tau(\psi, \phi) \in \mathcal{S}(\mathbb{R}^{2n})$ and the distributional bracket $\langle a, \text{Wig}_\tau(\psi, \phi) \rangle$ is thus well-defined. This defines \hat{A}_τ as a continuous operator $\mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$.

4.2 Averaging over τ

We define the (cross) Born–Jordan–Wigner (BJW) distribution of $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$ by the formula

$$\text{Wig}_{\text{BJ}}(\psi, \phi)(z) = \int_0^1 \text{Wig}_\tau(\psi, \phi) d\tau. \quad (42)$$

We set $\text{Wig}_{\text{BJ}} \psi = \text{Wig}_{\text{BJ}}(\psi, \psi)$. The properties of the BJW distribution are readily deduced from those of the τ -Wigner distribution studied above. In particular, the marginal properties (37) and (38) are obviously preserved:

$$\int_{\mathbb{R}^n} \text{Wig}_{\text{BJ}}(\psi, \phi)(z) dp = \psi(x) \overline{\phi}(x); \quad (43)$$

and

$$\int_{\mathbb{R}^n} \text{Wig}_{\text{BJ}}(\psi, \phi)(z) dx = \mathcal{F}\psi(p) \overline{\mathcal{F}\phi(p)}. \quad (44)$$

An object closely related to the (cross-)Wigner distribution is the (cross-)ambiguity function of a pair of functions $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$:

$$A(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p x'} \psi(x' + \frac{1}{2}x) \overline{\phi}(x' - \frac{1}{2}x) dx'.$$

It turns out that $A(\psi, \phi)$ and $\text{Wig}_\tau(\psi, \phi)$ are obtained from each other by a symplectic Fourier transform:

$$\text{Amb}(\psi, \phi) = \mathcal{F}_\sigma \text{Wig}(\psi, \phi), \quad \text{Wig}(\psi, \phi) = \mathcal{F}_\sigma \text{Amb}(\psi, \phi)$$

(\mathcal{F}_σ is an involution), thus justifying the following definition:

$$\text{Amb}_{\text{BJ}}(\psi, \phi) = \mathcal{F}_\sigma \text{Amb}(\psi, \phi).$$

An important property is that the BJW distribution of a function ψ is real, as is the usual Wigner distribution. In fact using the conjugacy formula

(36) we have

$$\begin{aligned}\overline{\text{Wig}_{\text{BJ}}\psi} &= \int_0^1 \overline{\text{Wig}_\tau\psi} d\tau = \int_0^1 \text{Wig}_{1-\tau}\psi d\tau \\ &= \int_0^1 \text{Wig}_\tau\psi d\tau = \text{Wig}_{\text{BJ}}\psi.\end{aligned}$$

Formula (41) relating Shubin's τ -operators to the τ -(cross) Wigner distribution carries over to the Born–Jordan case:

Corollary 5 *Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$, $A_{\text{BJ}} = \text{Op}_{\text{BJ}}(a)$. We have*

$$\langle A_{\text{BJ}}\psi|\bar{\phi} \rangle = \langle a, \text{Wig}_{\text{BJ}}(\psi, \phi) \rangle \quad (45)$$

for all $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. It suffices to integrate the equality $\langle A_\tau\psi|\phi \rangle = \langle a, \text{Wig}_\tau(\psi, \phi) \rangle$ with respect to $\tau \in [0, 1]$ and to use definitions (17) and (42). ■

One defines $A_{\text{BJ}} = \text{Op}_{\text{BJ}}(a)$ for arbitrary $a \in \mathcal{S}'(\mathbb{R}^{2n})$ by the same procedure as for Weyl operators and noting that $\text{Wig}_{\text{BJ}}(\psi, \phi) \in \mathcal{S}(\mathbb{R}^{2n})$ if $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$.

The following consequence of Proposition 4 will be essential in our study of the uncertainty principle:

Proposition 6 *Let Θ be defined by (27). We have*

$$\text{Wig}_{\text{BJ}}(\psi, \phi) = \text{Wig}(\psi, \phi) * \mathcal{F}\Theta \quad (46)$$

and

$$\text{Amb}_{\text{BJ}}(\psi, \phi) = (2\pi\hbar)^n \text{Amb}(\psi, \phi)\Theta \quad (47)$$

Proof. In view of formula (28) in Proposition 2 and formula (42) above we have

$$\langle a, \text{Wig}_{\text{BJ}}(\psi, \phi) \rangle = \langle b, \text{Wig}(\psi, \phi) \rangle$$

where $b = (2\pi\hbar)^n a * \mathcal{F}_\sigma\Theta$, hence

$$\begin{aligned}\langle a, \text{Wig}_{\text{BJ}}(\psi, \phi) \rangle &= (2\pi\hbar)^n \langle a * \mathcal{F}_\sigma\Theta, \text{Wig}(\psi, \phi) \rangle \\ &= (2\pi\hbar)^n \langle a, [\mathcal{F}_\sigma\Theta \circ (-I_d)] * \text{Wig}(\psi, \phi) \rangle \\ &= (2\pi\hbar)^n \langle a, \mathcal{F}_\sigma\Theta * \text{Wig}(\psi, \phi) \rangle\end{aligned}$$

(the last equality because Θ is an even function). Since $\mathcal{F}_\sigma\Theta = \mathcal{F}\Theta$ because Θ is even and invariant under permutation of the x and p variables, this proves formula (46), and formula (47) follows, taking the symplectic Fourier transform of both sides. ■

4.3 Symplectic (non-)covariance

The symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ is by definition the group of all linear automorphisms s of \mathbb{R}^{2n} which preserve the symplectic form $\sigma(z, z') = Jz \cdot z'$; equivalently $s^T J s = J$. The group $\mathrm{Sp}(2n, \mathbb{R})$ is a connected Lie group, and its double covering $\mathrm{Sp}_2(2n, \mathbb{R})$ has a faithful (but reducible) representation by a group of unitary operators, the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$ (see Folland [10], de Gosson [15, 16]). That group is generated by the operators \widehat{J} , $\widehat{M}_{L,m}$, and \widehat{V}_{-P} defined by $\widehat{J} = e^{-in\pi/4}\mathcal{F}$ and

$$\widehat{M}_{L,m}\psi(x) = i^m \sqrt{|\det L|} \psi(Lx), \quad \widehat{V}_P = e^{-\frac{i}{2\hbar}Px \cdot x} \psi(x)$$

where $L \in \mathrm{G}\ell(n, \mathbb{R})$ and $P \in \mathrm{Sym}(n, \mathbb{R})$; the integer m corresponds to the choice of an argument of $\det L$. Denoting by π^{Mp} the covering projection $\mathrm{Mp}(2n, \mathbb{R}) \longrightarrow \mathrm{Sp}(2n, \mathbb{R})$ we have $\pi^{\mathrm{Mp}}(\widehat{J}) = J$ and

$$\pi^{\mathrm{Mp}}(\widehat{M}_{L,m}) = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}, \quad \pi^{\mathrm{Mp}}(\widehat{V}_P) = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}.$$

Let now $A_W = \mathrm{Op}_W(a)$ be an arbitrary Weyl operator, and $s \in \mathrm{Sp}(2n, \mathbb{R})$. We have

$$\widehat{S} A_W \widehat{S}^{-1} = \mathrm{Op}_W(a \circ s^{-1}) \quad (48)$$

where $\widehat{S} \in \mathrm{Mp}(2n, \mathbb{R})$ is anyone of the two metaplectic operators such that $\pi^{\mathrm{Mp}}(\widehat{S}) = s$. This property is really characteristic of the Weyl correspondence; it is proven [15, 16] using the identity

$$\widehat{S} \widehat{T}(z) = \widehat{T}(sz) \widehat{S} \quad (49)$$

where $\widehat{T}(z)$ is the Heisenberg operator. One can show that if a pseudo-differential correspondence $a \longleftrightarrow \mathrm{Op}(a)$ (admissible in the sense above, or not) is such that $\mathrm{Op}(a \circ s^{-1}) = \widehat{S} \mathrm{Op}(a) \widehat{S}^{-1}$ then it *must* be the Weyl correspondence. For Born–Jordan correspondence we do still have a residual symplectic covariance, namely:

Proposition 7 *Let $A_{BJ} = \mathrm{Op}_{BJ}(a)$ with $a \in \mathcal{S}'(\mathbb{R}^{2n})$. We have*

$$\widehat{S} \mathrm{Op}_{BJ}(a) \widehat{S}^{-1} = \mathrm{Op}_{BJ}(a \circ s^{-1}) \quad (50)$$

for every \widehat{S} in the subgroup of $\mathrm{Mp}(2n, \mathbb{R})$ generated by the operators \widehat{J} and $\widehat{M}_{L,m}$ (with $\pi^{\mathrm{Mp}}(\widehat{S}) = s$).

Proof. (Cf. de Gosson [18]). It suffices to prove formula (50) for $\widehat{S} = \widehat{J}$ and $\widehat{S} = \widehat{M}_{L,m}$. Let first \widehat{S} be an arbitrary element of $\text{Mp}(2n, \mathbb{R})$; we have

$$\begin{aligned}\widehat{S} \text{Op}_{\text{BJ}}(a) &= \left(\frac{1}{2\pi\hbar}\right)^n \int a_\sigma(z) \Theta(z) \widehat{S} \widehat{T}(z) dz \\ &= \left(\frac{1}{2\pi\hbar}\right)^n \left(\int a_\sigma(z) \Theta(z) \widehat{T}(sz) dz \right) \widehat{S}\end{aligned}$$

where the second equality follows from the symplectic covariance property (49) of the Heisenberg operators. Making the change of variables $z' = sz$ in the integral we get, since $\det s = 1$,

$$\int a_\sigma(z) \Theta(z) \widehat{T}(sz) dz = \int a_\sigma(s^{-1}z) \Theta(s^{-1}z) \widehat{T}(z) dz.$$

Now, by definition of the symplectic Fourier transform we have

$$a_\sigma(s^{-1}z) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}\sigma(s^{-1}z, z')} a(z') dz' = (a \circ s^{-1})_\sigma(z).$$

Let now $\widehat{S} = \widehat{M}_{L,m}$; we have

$$\Theta(M_L^{-1}z) = \frac{\sin(Lp(L^T)^{-1}x/2\hbar)}{Lp(L^T)^{-1}x/2\hbar} = \Theta(z);$$

similarly $\Theta(J^{-1}z) = \Theta(z)$, hence in both cases

$$\begin{aligned}\widehat{S} \text{Op}_{\text{BJ}}(a) &= \left(\frac{1}{2\pi\hbar}\right)^n \left(\int (a \circ s^{-1})_\sigma \Theta(z) \widehat{T}(z) dz \right) \widehat{S} \\ &= \text{Op}_{\text{BJ}}(a \circ s^{-1}) \widehat{S}\end{aligned}$$

whence formula (50). ■

5 The Uncertainty Principle

We begin by reviewing the notion of density matrix (or operator) familiar from statistical quantum mechanics. The notion goes back to John von Neumann [32] in 1927, and is intimately related to the notion of mixed state (whose study mathematically belongs to the theory of C^* -algebras via the GNS construction). This will provide us with all the necessary tools for comparing the uncertainty relations in the Weyl and Born–Jordan case.

5.1 Density matrices

A density matrix on a Hilbert space \mathcal{H} is a self-adjoint positive operator on \mathcal{H} with trace one. In particular, it is a compact operator. Physically density matrices represent statistical mixtures of pure states, as explicitly detailed below.

We will need the two following results:

Lemma 8 *A self-adjoint trace class operator $\hat{\rho}$ on a Hilbert space \mathcal{H} is a density matrix if and only if there exists a sequence $(\alpha_j)_{j \geq 1}$ of positive numbers and a sequence of pairwise orthogonal finite-dimensional subspaces $(\mathcal{H}_j)_{j \geq 1}$ of \mathcal{H} such that*

$$\hat{\rho} = \sum_{j \geq 1} \alpha_j \Pi_j, \quad \sum_{j \geq 1} m_j \alpha_j = 1$$

where Π_j is the orthogonal projection of \mathcal{H} on \mathcal{H}_j and $m_j = \dim \mathcal{H}_j$

It is a consequence of the spectral decomposition theorem for compact operators (for a detailed proof see *e.g.* [16], §13.1).

Lemma 9 *Let $\psi \in L^2(\mathbb{R}^n)$, $\psi \neq 0$. The projection operator $P_\psi : L^2(\mathbb{R}^n) \rightarrow \{\lambda\psi : \lambda \in \mathbb{C}\}$ has Weyl and Born–Jordan symbols given by, respectively*

$$\rho_W = (2\pi\hbar)^n \text{Wig}_W \psi \quad (51)$$

and

$$\rho_{BJ} = (2\pi\hbar)^n \text{Wig}_{BJ} \psi. \quad (52)$$

In particular ρ_W and ρ_{BJ} are real functions.

Proof. We have $P_\psi \phi = (\phi|\psi)_{L^2} \psi$ hence the kernel of P_ψ is $K_\psi = \psi \otimes \overline{\psi}$. Using a Fourier transform formula (21) implies that the τ -symbol ρ_τ of P_ψ is given by

$$\begin{aligned} \rho_\tau(z) &= \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p y} K_\psi(x + \tau y, x - (1 - \tau)y) dy \\ &= \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p y} \psi(x + \tau y) \overline{\psi}(x - (1 - \tau)y) dy \\ &= (2\pi\hbar)^n \text{Wig}_\tau \psi(z). \end{aligned}$$

Setting $\tau = \frac{1}{2}$ we get formula (51). Formula (52) is obtained by integrating $\rho_\tau(z)$ with respect to $\tau \in [0, 1]$. That ρ_W and ρ_{BJ} are real follows from the fact that both the Wigner and the WBJ distribution are real. ■

The following result describes density matrices in both the Weyl and Born–Jordan case in terms of the Wigner formalism:

Proposition 10 *Let $\hat{\rho}$ be a density matrix on $L^2(\mathbb{R}^n)$. There exists an orthonormal system $(\psi_j)_{j \geq 1}$ of $L^2(\mathbb{R}^n)$ and a sequence of non-negative numbers $(\lambda_j)_{j \geq 1}$ such that $\sum_{j \geq 1} \lambda_j = 1$ and*

$$\hat{\rho} = \text{Op}_{\text{BJ}} \rho_{\text{BJ}} = \text{Op}_{\text{W}} \rho_{\text{W}} \quad (53)$$

the symbols ρ_{W} and ρ_{BJ} being given by

$$\rho_{\text{W}} = (2\pi\hbar)^n \sum_{j \geq 1} \lambda_j \text{Wig} \psi_j \quad (54)$$

and

$$\rho_{\text{BJ}} = (2\pi\hbar)^n \sum_{j \geq 1} \lambda_j \text{Wig}_{\text{BJ}} \psi_j. \quad (55)$$

Proof. Taking $\mathcal{H} = L^2(\mathbb{R}^n)$ in Lemma 8 we can write $\hat{\rho} = \sum_j \alpha_j \Pi_j$ where each Π_j is the projection operator on a finite dimensional space $\mathcal{H}_j \subset L^2(\mathbb{R}^n)$, and two spaces \mathcal{H}_j and \mathcal{H}_ℓ are orthonormal if $j \neq \ell$. For each index j let us choose an orthonormal basis $\mathcal{B}_j = (\psi_{j+1}, \dots, \psi_{j+m_j})$ of \mathcal{H}_j ; the union $\mathcal{B} = \cup_j \mathcal{B}_j$ is then an orthonormal basis of $\oplus_j \mathcal{H}_j$, and we have, using Lemma 8

$$\hat{\rho} = \sum_{j \geq 1} \alpha_j \left(\sum_{j+1 \leq k \leq j+m_j} \Pi_{\psi_k} \right)$$

where Π_{ψ_k} is the orthogonal projection on the ray $\{\lambda \psi_j : \lambda \in \mathbb{C}\}$. Since each index α_j is repeated m_j times due to the expression between brackets, this can be rewritten

$$\hat{\rho} = \sum_{j \geq 1} m_j \alpha_j \Pi_{\psi_j} = \sum_{j \geq 1} \lambda_j \Pi_{\psi_j}$$

with the $\lambda_j = m_j \alpha_j$ summing up to one. In view of Lemma 9 the Weyl (resp. Born–Jordan) symbol of Π_{ψ_j} is $(2\pi\hbar)^n \text{Wig}_{\text{W}} \psi$ (resp. $(2\pi\hbar)^n \text{Wig}_{\text{BJ}} \psi$) hence the result. ■

Notice that the orthonormal bases \mathcal{B}_j in the proof can be chosen arbitrarily; the decompositions (54) and (55) are therefore not unique. (In Physics, one would say that a mixed quantum state can be written in infinitely many way as a superposition of pure states, a pure state being a density operator with symbol a Wigner function).

5.2 A general uncertainty principle

In what follows the notation

$$a \longleftrightarrow \hat{A} = \text{Op}(a)$$

is indifferently the Weyl or the Born–Jordan correspondence. Both have the property:

If the symbol a is real, then the operator \hat{A} is essentially self-adjoint (in which case we call it an observable).

Notice that the Shubin correspondence does not have this property for $\tau \neq \frac{1}{2}$ since $A_\tau^* = A_{1-\tau}$.

Let $\hat{\rho}$ be a density matrix on $L^2(\mathbb{R}^n)$. We assume that $\hat{\rho} = (2\pi\hbar)^n \text{Op}(\rho)$. The function ρ is a real function on phase space \mathbb{R}^{2n} and we have

$$\text{Tr } \hat{\rho} = \int_{\mathbb{R}^{2n}} \rho(z) dz = 1. \quad (56)$$

Observe that we do not in general have $\rho \geq 0$.

Let \hat{A} be an observable. Its expectation value with respect to ρ is by definition the real number

$$\langle \hat{A} \rangle = \int_{\mathbb{R}^{2n}} a(z) \rho(z) dz \quad (57)$$

where it is assumed that the integral on the right side is absolutely convergent. We will write, with some abuse of notation,

$$\langle \hat{A} \rangle = \text{Tr}(\rho \hat{A}) \quad (58)$$

(see the discussion in de Gosson [16], §12.3, of the validity of various “trace formulas”). In view of formulas (40) and (45) we have either

$$\langle \hat{A} \rangle = (2\pi\hbar)^n \sum_{j \geq 1} \lambda_j \langle a, \text{Wig } \psi_j \rangle$$

(when $a \longleftrightarrow \hat{A}$ is the Weyl correspondence) or

$$\langle \hat{A} \rangle = (2\pi\hbar)^n \sum_{j \geq 1} \lambda_j \langle a, \text{Wig}_{\text{BJ}} \psi_j \rangle$$

(when $a \longleftrightarrow \hat{A}$ is the Born–Jordan correspondence) and hence

$$\langle \hat{A} \rangle = (2\pi\hbar)^n \sum_{j \geq 1} \lambda_j (\hat{A} \psi_j | \psi_j)_{L^2} = \sum_{j \geq 1} \lambda_j \langle \hat{A} \rangle_j \quad (59)$$

where $\langle \hat{A} \rangle_j = (\hat{A} \psi_j | \psi_j)_{L^2}$ (recall that $(\psi_j)_{j \geq 1}$ is an orthonormal system).

If \hat{A}^2 also is an observable and if $\langle \hat{A}^2 \rangle = \text{Tr}(\rho \hat{A}^2)$ exists, then the number

$$(\text{Var}_\rho \hat{A})^2 = \langle \hat{A}^2 \rangle_\rho - \langle \hat{A} \rangle_\rho^2 \quad (60)$$

that is

$$(\text{Var}_\rho \hat{A})^2 = \text{Tr}(\hat{\rho} \hat{A}^2) - \text{Tr}(\hat{\rho} \hat{A})^2 \quad (61)$$

is the variance of \hat{A} ; its positive square root $\text{Var}_\rho \hat{A}$ is called “standard deviation”. More generally consider a second observable \hat{B} ; then the covariance of the pair (\hat{A}, \hat{B}) with respect to $\hat{\rho}$ is defined by

$$\text{Cov}_\rho(\hat{A}, \hat{B}) = \text{Tr}(\hat{\rho} \hat{A} \hat{B}) - \text{Tr}(\hat{\rho} \hat{A}) \text{Tr}(\hat{\rho} \hat{B}). \quad (62)$$

It is in general a complex number, and we have

$$\overline{\text{Cov}_\rho(\hat{A}, \hat{B})} = \text{Cov}_\rho(\hat{B}, \hat{A}). \quad (63)$$

The covariance has the properties of a complex scalar product; it therefore satisfies the Cauchy–Schwarz inequality

$$\text{Tr}(\hat{\rho} \hat{A}^2) \text{Tr}(\hat{\rho} \hat{B}^2) \geq |\text{Cov}_\rho(\hat{A}, \hat{B})|^2. \quad (64)$$

The following lemma will be useful in the proof of the uncertainty inequalities below:

Lemma 11 *If the covariance of two observables \hat{A} and \hat{B} exist we have*

$$\text{Re Cov}_\rho(\hat{A}, \hat{B}) = \frac{1}{2} \text{Tr}(\hat{\rho} \{\hat{A}, \hat{B}\}) - \text{Tr}(\hat{\rho} \hat{A}) \text{Tr}(\hat{\rho} \hat{B}) \quad (65)$$

$$\text{Im Cov}_\rho(\hat{A}, \hat{B}) = \frac{1}{2i} \text{Tr}(\hat{\rho} [\hat{A}, \hat{B}]) \quad (66)$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ and $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ are, respectively, the anticommutator and the commutator of \hat{A} and \hat{B} .

Proof. We have, in view of (63),

$$\begin{aligned} 2 \operatorname{Re} \operatorname{Cov}_\rho(\hat{A}, \hat{B}) &= \operatorname{Cov}_\rho(\hat{A}, \hat{B}) + \operatorname{Cov}_\rho(\hat{B}, \hat{A}) \\ &= \operatorname{Tr}(\hat{\rho}\{\hat{A}, \hat{B}\}) - 2 \operatorname{Tr}(\hat{\rho}\hat{A}) \operatorname{Tr}(\hat{\rho}\hat{B}); \end{aligned}$$

for the second equality

$$\begin{aligned} 2i \operatorname{Im} \operatorname{Cov}_\rho(\hat{A}, \hat{B}) &= \operatorname{Cov}_\rho(\hat{A}, \hat{B}) - \operatorname{Cov}_\rho(\hat{B}, \hat{A}) \\ &= \operatorname{Tr}(\hat{\rho}\hat{A}\hat{B}) - \operatorname{Tr}(\hat{\rho}\hat{B}\hat{A}) \\ &= \operatorname{Tr}(\hat{\rho}[\hat{A}, \hat{B}]). \end{aligned}$$

■

Observe that the anticommutator and commutator obey the relations

$$\{\hat{A}, \hat{B}\}^* = \{\hat{A}, \hat{B}\} \quad , \quad [\hat{A}, \hat{B}]^* = -[\hat{A}, \hat{B}]$$

and hence $\langle[\hat{A}, \hat{B}]\rangle$ is a pure imaginary number or zero; in particular $|\langle[\hat{A}, \hat{B}]\rangle|^2 \leq 0$.

Proposition 12 *If the variances and covariance of two observables \hat{A} and \hat{B} exist then:*

$$(\operatorname{Var}_\rho \hat{A})^2 (\operatorname{Var}_\rho \hat{B})^2 \geq \operatorname{Cov}_\rho^{\operatorname{sym}}(\hat{A}, \hat{B})^2 - \frac{1}{4} \langle[\hat{A}, \hat{B}]\rangle_\rho^2 \quad (67)$$

where $\langle[\hat{A}, \hat{B}]\rangle^2 < 0$ and

$$\operatorname{Cov}_\rho^{\operatorname{sym}}(\hat{A}, \hat{B}) = \frac{1}{2} (\operatorname{Cov}_\rho(\hat{A}, \hat{B}) + \operatorname{Cov}_\rho(\hat{B}, \hat{A})) \quad (68)$$

is a real number. In particular the Heisenberg inequality

$$(\operatorname{Var}_\rho \hat{A})^2 (\operatorname{Var}_\rho \hat{B})^2 \geq -\frac{1}{4} \langle[\hat{A}, \hat{B}]\rangle_\rho^2 \quad (69)$$

holds.

Proof. Replacing \hat{A} and \hat{B} with $\hat{A} - \langle\hat{A}\rangle$ and $\hat{B} - \langle\hat{B}\rangle$ it is sufficient to prove (67) when $\langle\hat{A}\rangle = \langle\hat{B}\rangle = 0$. We thus have to prove the inequality

$$\operatorname{Tr}(\hat{\rho}\hat{A}^2) \operatorname{Tr}(\hat{\rho}\hat{B}^2) \geq \operatorname{Cov}_\rho^{\operatorname{sym}}(\hat{A}, \hat{B})^2 - \frac{1}{4} \operatorname{Tr}(\hat{\rho}[\hat{A}, \hat{B}])^2. \quad (70)$$

Noting that definition (68) can be rewritten

$$\operatorname{Cov}_\rho^{\operatorname{sym}}(\hat{A}, \hat{B}) = \frac{1}{2} \operatorname{Tr}(\hat{\rho}\{\hat{A}, \hat{B}\}) - \operatorname{Tr}(\hat{\rho}\hat{A}) \operatorname{Tr}(\hat{\rho}\hat{B}).$$

we remark that in view of formulas (65) and (66) in the lemma above we have

$$|\text{Cov}_\rho(\hat{A}, \hat{B})|^2 = \frac{1}{4} \text{Tr}(\hat{\rho}\{\hat{A}, \hat{B}\})^2 - \frac{1}{4} \text{Tr}(\hat{\rho}[\hat{A}, \hat{B}])^2 \quad (71)$$

$$= \text{Cov}_\rho^{\text{sym}}(\hat{A}, \hat{B})^2 - \frac{1}{4} \text{Tr}(\hat{\rho}[\hat{A}, \hat{B}])^2 \quad (72)$$

hence the proof of (70) is reduced to the proof of the inequality

$$\text{Tr}(\hat{\rho}\hat{A}^2) \text{Tr}(\hat{\rho}\hat{B}^2) \geq |\text{Cov}_\rho(\hat{A}, \hat{B})|^2 \quad (73)$$

which is just the Cauchy–Schwarz inequality (64) for covariances. ■

5.3 Weyl vs Born–Jordan

Let us discuss the similarities and differences between the uncertainty principles associated with the Weyl and Born–Jordan correspondences. First, as already observed in the Introduction, the Weyl and Born–Jordan quantizations of monomials $x_j^m p_j^n$ (and hence of their linear combinations) are identical when $m + n \leq 2$. This implies, in particular, that if the symbols a and b are, respectively, multiplication by the coordinates x_j and p_j then the corresponding operators \hat{A} and \hat{B} are, in both cases given by $\hat{X}_j = x_j$ and $\hat{P}_j = -i\hbar\partial_{x_j}$. It follows that $\text{Var}_\rho \hat{X}_j$ and $\text{Var}_\rho \hat{P}_j$ satisfy the usual Robertson–Schrödinger inequalities

$$\text{Var}_\rho \hat{X}_j \text{Var}_\rho \hat{P}_j \geq \text{Cov}_\rho^{\text{sym}}(\hat{X}_j, \hat{P}_j)^2 + \frac{1}{4}\hbar^2. \quad (74)$$

We mention that the inequalities (74) can be rewritten in compact form as

$$\Sigma + \frac{1}{2}i\hbar J \geq 0 \quad (75)$$

where ≥ 0 means “semi-definite positive”, J is the standard symplectic matrix, and

$$\Sigma = \begin{pmatrix} \text{Cov}_\rho(\hat{X}, \hat{X}) & \text{Cov}_\rho(\hat{X}, \hat{P}) \\ \text{Cov}_\rho(\hat{P}, \hat{X}) & \text{Cov}_\rho(\hat{P}, \hat{P}) \end{pmatrix} \quad (76)$$

is the statistical covariance matrix. The formulation (75) of the Robertson–Schrödinger inequalities clearly shows one of the main features, namely the symplectic covariance of these inequalities, which we have used in previous work [17, 20] to express the uncertainty principle in terms of the notion of symplectic capacity, which is closely related to Gromov’s non-squeezing theorem from symplectic topology. This has also given us the opportunity to discuss the relations between classical and quantum mechanics in [19].

Let $\widehat{S} \in \text{Mp}(2n, \mathbb{R})$, $S = \pi^{\text{Mp}}(\widehat{S})$ and set $\widehat{A}' = \widehat{S}\widehat{A}\widehat{S}^{-1}$, $\widehat{B}' = \widehat{S}\widehat{B}\widehat{S}^{-1}$; we are assuming that \widehat{A}, \widehat{B} correspond, as in the proof of Proposition 12, to an arbitrary quantization scheme $a \longleftrightarrow \widehat{A}$. We have quite generally, using the cyclicity of the trace,

$$\begin{aligned} \text{Cov}_{\rho}^{\text{sym}}(\widehat{A}, \widehat{B}) &= \frac{1}{2} \text{Tr}(\widehat{\rho}\{\widehat{A}, \widehat{B}\}) - \text{Tr}(\widehat{\rho}\widehat{A}) \text{Tr}(\widehat{\rho}\widehat{B}) \\ &= \frac{1}{2} \text{Tr}(\widehat{\rho}\widehat{S}^{-1}\{\widehat{A}', \widehat{B}'\}\widehat{S}) - \text{Tr}(\widehat{\rho}\widehat{S}^{-1}\widehat{A}'\widehat{S}) \text{Tr}(\widehat{\rho}\widehat{S}^{-1}\widehat{B}'\widehat{S}) \\ &= \frac{1}{2} \text{Tr}(\widehat{S}^{-1}\widehat{\rho}\widehat{S}\{\widehat{A}', \widehat{B}'\}) - \text{Tr}(\widehat{S}^{-1}\widehat{\rho}\widehat{S}\widehat{A}') \text{Tr}(\widehat{S}^{-1}\widehat{\rho}\widehat{S}\widehat{B}') \\ &= \text{Cov}_{\widehat{S}\widehat{\rho}\widehat{S}^{-1}}^{\text{sym}}(\widehat{A}', \widehat{B}'); \end{aligned}$$

similarly $\text{Var}_{\rho} \widehat{A} = \text{Var}_{\widehat{S}\widehat{\rho}\widehat{S}^{-1}} \widehat{A}$ and $\langle [\widehat{A}, \widehat{B}] \rangle_{\rho}^2 = \langle [\widehat{A}', \widehat{B}'] \rangle_{\widehat{S}\widehat{\rho}\widehat{S}^{-1}}^2$ and hence

$$(\text{Var}_{\rho'} \widehat{A}')^2 (\text{Var}_{\rho'} \widehat{B}')^2 \geq \text{Cov}_{\rho'}^{\text{sym}}(\widehat{A}', \widehat{B}')^2 - \frac{1}{4} \langle [\widehat{A}', \widehat{B}'] \rangle_{\rho'}^2.$$

with $\rho' = \widehat{S}\widehat{\rho}\widehat{S}^{-1}$. Suppose now that the operator correspondence $a \longleftrightarrow \widehat{A}$ is the Weyl correspondence; then, by Proposition 7 we have $\widehat{S}\widehat{\rho}\widehat{S}^{-1} = \text{Op}(\rho \circ s^{-1})$ and the inequalities (67) become

$$(\text{Var}_{\rho \circ s^{-1}} A'_W)^2 (\text{Var}_{\rho \circ s^{-1}} B'_W)^2 \geq \text{Cov}_{\rho \circ s^{-1}}^{\text{sym}}(A'_W, B'_W)^2 - \frac{1}{4} \langle [A'_W, B'_W] \rangle_{\rho \circ s^{-1}}^2. \quad (77)$$

Again, in view of Proposition 7, in the Born–Jordan case we have inequality

$$(\text{Var}_{\rho \circ s^{-1}} A'_{\text{BJ}})^2 (\text{Var}_{\rho \circ s^{-1}} B'_{\text{BJ}})^2 \geq \text{Cov}_{\rho \circ s^{-1}}^{\text{sym}}(A'_{\text{BJ}}, B'_{\text{BJ}})^2 - \frac{1}{4} \langle [A'_{\text{BJ}}, B'_{\text{BJ}}] \rangle_{\rho \circ s^{-1}}^2 \quad (78)$$

only for those $\widehat{S} \in \text{Mp}(2n, \mathbb{R})$ which are products of metaplectic operators of the type \widehat{J} and $\widehat{M}_{L,m}$.

6 Discussion

There is an old ongoing debate in quantum mechanics on which quantization scheme is the most adequate for physical applications; an interesting recent contribution is that of Kauffmann [25], who seems to favor the Born–Jordan correspondence. The introduction of the τ -Wigner and Born–Jordan distributions has been motivated in time-frequency analysis by the fact that the usual cross-Wigner distribution gives raise to disturbing ghost frequencies; it was discovered by Boggiatto and his collaborators [1, 2, 3] that these ghost frequencies were attenuated by averaging over τ .

The study of uncertainties for non-standard situations has been tackled (from a very different point of view) by Korn [26]; also see the review paper [11] by Folland and Sitaram, which however unfortunately deliberately

ignores the fundamental issue of covariance. Gibilisco and his collaborators [12, 13, 14] give highly nontrivial refinements of uncertainty relations using convexity properties, and studied the notion of statistical covariance in depth.

We mention that in a very well written thesis, published as a book, Steiger [31] has given an interesting historical review and analysis of the evolution of the uncertainty principle; in addition he compares the interest of several different formulations, and gives a clever elementary derivation of the Robertson–Schrödinger inequalities for operators. The work also contains Mathematica codes for the computation of (co-)variances.

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References

- [1] P. Boggiatto, G. De Donno, A. Oliaro, Time-Frequency Representations of Wigner Type and Pseudo-Differential Operators, T. Am. Math. Soc. 362(9) (2010) 4955–4981.
- [2] P. Boggiatto, Bui Kien Cuong, G. De Donno, A. Oliaro, Weighted integrals of Wigner representations, J.Pseudo-Differ. Oper. Appl. 1(4) (2010) 401–415.
- [3] P. Boggiatto, G. De Donno, A. Oliaro. Hudson Theorem for τ -Wigner Transforms, Quaderni scientifici del Dipartimento di Matematica, Università di Torino, Quaderno N. 1 (2010).
- [4] M. Born, P. Jordan, Zur Quantenmechanik, Z. Physik 34, 858–888 (1925).
- [5] M. Born, W. Heisenberg, P. Jordan, Zur Quantenmechanik II, Z. Physik 35 (1926) 557–615.
- [6] L. Castellani, Quantization Rules and Dirac’s Correspondence, Il Nuovo Cimento 48A(3) (1978) 359–368.
- [7] P. Crehan, The parametrisation of quantisation rules equivalent to operator orderings, and the effect of different rules on the physical spectrum, J. Phys. A: Math. Gen. 22 (1989) 811–822.
- [8] P.A.M. Dirac, The fundamental equations of quantum mechanics, Proc. R. Soc. London, Ser. A, 109 (1925) 642–653.

- [9] W.A. Fedak, J.J. Prentis, The 1925 Born and Jordan paper “On quantum mechanics”, *Am. J. Phys.* 77(2) (2009) 128–139.
- [10] G.B. Folland, *Harmonic Analysis in Phase space*, *Annals of Mathematics studies*, Princeton University Press, Princeton, N.J., 1989.
- [11] G.B. Folland, A. Sitaram, The uncertainty principle: a mathematical survey, *J. Fourier Anal. Appl.* 3(3) (2007) 207–238
- [12] P. Gibilisco, D. Imparato, and T. Isola, Uncertainty principle and quantum Fisher information II, *J. Math. Phys.* 48, 072109 (2007).
- [13] P. Gibilisco, F. Hiai, D. Petz, Quantum covariance, quantum Fisher information, and the uncertainty relations, *IEEE Trans. Inform. Theory* 55(1) (2009) 439–443.
- [14] P. Gibilisco, T. Isola, How to distinguish quantum covariances using uncertainty relations. *J. Math. Anal. Appl.* 384(2) (2011) 2, 670–676
- [15] M. de Gosson, *Symplectic Geometry and Quantum Mechanics*, Birkhäuser, Basel, 2006.
- [16] M. de Gosson, *Symplectic Methods in Harmonic Analysis; Applications to Mathematical Physics*, Birkhäuser, 2011.
- [17] M. de Gosson, The symplectic camel and the uncertainty principle: the tip of an iceberg? *Found. Phys.* 39(2) (2009) 194–214
- [18] M. de Gosson, Symplectic Covariance Properties for Shubin and Born–Jordan Pseudo-Differential Operators. *T. Am. Math. Soc.* (2012).
- [19] M. de Gosson, B. Hiley, Imprints of the Quantum World in Classical Mechanics, *Found. Phys.* 41(9) (2011) 1415–1436.
- [20] M. de Gosson, F. Luef, Symplectic capacities and the geometry of uncertainty: the irruption of symplectic topology in classical and quantum mechanics. *Phys. Rep.* 484(5) (2009) 131–179.
- [21] M. de Gosson, F. Luef, Preferred Quantization Rules: Born–Jordan vs. Weyl; Applications to Phase Space Quantization. *J.Pseudo-Differ. Oper. Appl.* 2(1) (2011) 115–139.
- [22] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2000.

- [23] W. Heisenberg, Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen, *Zeitschrift für Physik*, 33 (1925) 879–893; English translation in: B.L. van der Waerden, editor, *Sources of Quantum Mechanics*, Dover Publications, 1968.
- [24] R.L. Hudson, When is the Wigner quasi-probability density non-negative? *Rep. Math. Phys.* 6 (1974), 249–252.
- [25] S.K. Kauffmann, Unambiguous quantization from the maximum classical correspondence that is self-consistent: the slightly stronger canonical commutation rule Dirac missed, *Found. Phys.* 41 (2011) 805–918.
- [26] P. Korn, Some Uncertainty Principles for Time-Frequency Transforms of the Cohen Class, *IEEE T. Signal. Proces.* 53(2) (2005) 523–527
- [27] N.H. McCoy, On the function in quantum mechanics which corresponds to a given function in classical mechanics, *Proc. Natl. Acad. Sci. U.S.A.* 18(11) (1932) 674–676.
- [28] E. Scholz, Weyl entering the ‘new’ quantum mechanics discourse, C. Joas, C. Lehner, J. Renn (eds.). *HQ-1: Conference on the History of Quantum Physics* (Berlin, July 2–6, 2007), Preprint MPI History of Science Berlin, 350 vol. II (2007).
- [29] E. Schrödinger, Zum Heisenbergschen Unschärfeprinzip, *Sitzungsberichte der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse* (1930) 296–303.
- [30] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, 1987; original Russian edition in Nauka, Moskva, 1978.
- [31] N.J. Steiger, Quantum Uncertainty and Conservation Law Restrictions on Gate Fidelity, Brigham Young University. Department of Physics and Astronomy, 2010.
- [32] J. von Neumann, Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik, *Göttinger Nachrichten* 1 (1927) 245–272.
- [33] H. Weyl, Quantenmechanik und Gruppentheorie, *Zeitschrift für Physik*, 46 (1927).
- [34] H. Weyl, *The Theory of Groups and Quantum Mechanics*, translated from the 2nd German edition by H.P. Robertson, New York, Dutton (1931).